Finite horizon model predictive control with ellipsoid mapping of uncertain linear systems

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Abstract: A model predictive control scheme was proposed for discrete-time uncertain linear systems subject to input constraints. The cost functional to be minimised is a finite horizon quadratic cost, which describes the performance of the corresponding nominal system. The control action is specified in terms of both feedback and open-loop components. The open-loop part of the control action steers the centre of associated ellipsoids into a set around the origin, while the feedback component forces the actual system states to remain in those ellipsoids. Both feedback and open-loop control are determined online by repeatedly solving a convex optimisation problem. The predictive control scheme guarantees recursive feasibility and robust stability if the convex optimisation problem is feasible at the initial time instant. A numerical example illustrates the effectiveness of the proposed approach.

Nomenclature

The following notations are used throughout the paper. Let \( \mathbb{R} \) and \( \mathbb{Z} \) denote the field of real number, the set of integer numbers, \( \mathbb{R}^n \) denotes the n-dimensional Euclidean space.

The notations \( \mathbb{Z}_{\{c_1,c_2\}} \) and \( \mathbb{Z}_{\{c_1,c_2\}}^+ \) are to denote the sets \( \{k \in \mathbb{Z} \mid c_1 \leq k \leq c_2\} \) and \( \{k \in \mathbb{Z} \mid c_1 \leq k < c_2\} \). For a matrix \( M \in \mathbb{R}^{n \times n} \), \( M^\top \) denotes the transpose of \( M \) and \( \sigma(M) \) denotes the largest singular value of matrix \( M \). \( I_m \) denotes the \( m \times m \) identity matrix, and \( \ast \) denotes the corresponding symmetric block in symmetric matrices.

1 Introduction

Model predictive control (MPC) has received remarkable attention in both practical applications and theoretical research over the last 30 years since it yields optimal performance and is capable of explicitly dealing with state and input constraints. The basic idea of standard MPC [1–4] is as follows: online, a finite horizon open-loop optimal control problem based on the current measurement of the system states is solved. Then, the first part of the obtained open-loop optimal input trajectory is applied to the system. At the succeeding time instant, the optimal control problem is solved again using new state measurements, and the actual control input is updated.

However, for a nominally stabilising MPC scheme with the presence of disturbances and/or model uncertainties might lead to performance deterioration or even loss of stability [5]. This basically results from two major problems of standard MPC. First, the solution to the optimal control problem is open-loop trajectory, and feedback is only provided at the sampling instants [6]. Second, recursive feasibility often cannot be guaranteed for all admissible uncertainty realisations [7].

An intuitive approach to guarantee robust stability and recursive feasibility is to use a min-max MPC formulation, where the optimal input is determined such that the performance criteria is minimised for a worst-case uncertainty [8–14]. However, such approaches are computationally expensive in general. Furthermore, the optimal input is obtained for a possibly unrealistic worst-case scenario, which often results in poor performance in the case of small actual uncertainties.

For uncertain linear systems, the min–max MPC formulation is circumvented in [15] by repeatedly solving a semi-definite program (SDP) such that an upper bound on the worst-case performance is minimised. This computationally attractive approach is based on the online calculation of robustly positively invariant ellipsoids and associated feedback matrices. The price to pay is a rather small region of attraction.

Many research activities focused on enlarging the region of attraction and/or improving control performance while keeping the computational burden as low as possible.

Using parameter-dependent Lyapunov functions, the authors of [16–18] propose MPC schemes that guarantee asymptotic stability rather than exponential stability, and provide extra degrees of freedom to reduce the conservative of the optimisation problem.

A fixed state-feedback law with perturbations is proposed in [19], where the system trajectory tracks the trajectory related to an a priori fixed state-feedback control law.

A parameter-dependent feedback law in the framework of gain-scheduling is proposed in [20], which offers potential
performance improvements compared with approaches with static feedback laws.

By allowing the first control action to be chosen freely, the robust MPC schemes in [21–23] are applicable to systems subject to asymmetric constraints.

Linear parameter-varying (LPV) system is a particular case of linear uncertain systems whose dynamics depend on time-varying parameters. The MPC scheme of LPV systems in [24] is based on ellipsoid mapping over a finite horizon. This approach requires that the rate of the parameter variation is bounded, and is thus restricted to systems with slowly varying parameters. Recently, robust MPC schemes for linear systems with structured feedback uncertainty have been exploited by Smith [25, 26]. The control law is specified in terms of both feedback and open-loop components. The open-loop part steers the trajectory of the nominal system to the origin at the end of the prediction horizon [25] or at some time instants within the prediction horizon [26], whereas the associated feedback law renders some prescribed ellipsoids invariants.

In this paper, we introduce a finite horizon robust MPC scheme of linear systems to subject feedback uncertainty and input constraints. The considered finite horizon cost functional, including a terminal function penalising the state at the end of the prediction horizon, solely depends on the trajectories of the nominal system, which is different from the results proposed by the authors of [15, 20, 21], where a worst-case functional is minimised. Similar to [25, 26], the idea is to divide the control law in both feedback and open-loop components. The open-loop component steers the nominal trajectories into a nominal terminal set around the origin. The feedback component ensures that the actual state trajectories remain in some associated predicted ellipsoids for any admissible uncertainty realisation. The ellipsoids are calculated such that any perturbed state at the end of the prediction horizon lies in the actual terminal set, which entirely contains the nominal terminal set. The terminal penalty function represents an upper bound on the infinite horizon cost obtained by fictitiously applying a linear terminal feedback law, which renders the actual terminal region to be robust positive invariant. The resulting online optimisation problem is formulated as a convex optimisation problem, and the proposed scheme guarantees robust stability and recursive feasibility if the optimisation problem is initially feasible. The terminal set, terminal penalty function, and terminal feedback law are determined by solving linear matrix inequalities (LMIs) offline.

The paper is structured as follows: Section 2 introduces the system considered, and presents some results on ellipsoid mapping, constraints satisfaction and the choice of design parameters. In Section 3, the novel robust MPC scheme is proposed, together with a discussion of its recursive feasibility and asymptotic stability properties. Simulation results are reported in Section 4.

To derive the results proposed in this section, some preliminary results are used. First, we consider the well-known S-procedure.

Lemma 1 (S-procedure for quadratic functions) [27]: Let $F_0, F_1, \ldots, F_p$ be quadratic functions of the variable $\xi \in \mathbb{R}^n$

$$F_i(\xi) := \xi^T T_i \xi + 2 \beta_i^T \xi + \delta_i, \quad i \in \mathbb{Z}_{(0,p)}$$

where $\beta_i \in \mathbb{R}^n$, $T_i \in \mathbb{R}^{n \times n}$ and $\delta_i$ is a scalar. We consider the following condition on $F_0, \ldots, F_p$

$$F_0(\xi) \geq 0, \quad \text{for all } \xi \text{ such that } F_i(\xi) \geq 0, \quad i \in \mathbb{Z}_{(1,p)}$$  (1)

If there exist $\tau_i > 0$, for all $i \in \mathbb{Z}_{(1,p)}$, such that for all $\xi$

$$F_i(\xi) - \sum_{i=0}^{p} \tau_i F_i(\xi) \geq 0$$

then (1) holds. For $p = 1$, the converse holds, provided that there is some $\xi_0$ such that $F_1(\xi_0) \geq 0$.

We further require the following result.

Lemma 2 [28, 29]: Let $G \in \mathbb{R}^{n \times n}$, $H \in \mathbb{R}^n$, $x \in \mathbb{R}^n$ and $c \in \mathbb{R}$. The inequality

$$x^T G x + 2 H^T x + c \leq 0$$  (2)

is satisfied for all $x \in \mathbb{R}^n$, if and only if

$$\begin{bmatrix} G & H^T \\ H & c \end{bmatrix} \leq 0$$  (3)

2 Problem setup

Consider the discrete-time linear system with structured feedback uncertainty; see Fig. 1.

$$x_{k+1} = Ax_k + Bu_k + B_p p_k$$  (4a)

$$q_k = C_p x_k + D_p u_k$$  (4b)

$$p_k = \Delta_p q_k$$  (4c)

where $x_k \in \mathbb{R}^{n_k}$ is the state, $u_k \in U \subseteq \mathbb{R}^{n_u}$ is the control input, and $p_k \in \mathbb{R}^{n_p}$ and $q_k \in \mathbb{R}^p$ describe the structured feedback uncertainty of the considered system. The input constraint set is defined as

$$U := \{u \in \mathbb{R}^{n_u} \mid g_k^T u \leq h_j, \quad j \in \mathbb{Z}_{(1,n_u)}\}$$  (5)

where $r_u \in \mathbb{Z}_{(0,\infty)}$ is the number of input constraints, $g_k \in \mathbb{R}^{n_u \times 1}$ and $h_j \in \mathbb{R}$. Denote

$$\Xi := \{\Delta \in \mathbb{R}^{n_u \times n} \mid \Delta := \text{diag} \{\Delta_1, \Delta_2, \ldots, \Delta_m\}, \quad \Delta_i \preceq 1, \quad \Delta_i \in \mathbb{R}^{n_u \times n}, \quad l \in \mathbb{Z}_{(1,m)}, \sum_{l=1}^{m} n_l = n_p\}$$  (6)

where $\Delta_i$ is a repeated scalar or a full block. $\Delta_i$ models a number of factors, such as non-linearities, dynamics or
parameters, that are unknown, unmodelled or neglected. The
operator Δi ∈ ℤ is a block diagonal matrix.

Denote the projection onto the ℓth component associated
with Δi as Πi, that is, Πi satisfies Δi = ΠiΔ, Δ ∈ ℤ. Then,
the norm bound on each Δi implies that
\[
(Πiρi)^TΠiρi \leq (Πiρi)^TΠiρi, \quad l \in ℤ[1,n]
\]
(7)

Remark 1: The uncertainty formulation can also be viewed
as replacing the state-space matrices (A, B) by (A, B) := \{A + BΔC, B + BΔD\}, where (A, B) := \{A + BΔC, B + BΔD\}, Δ ∈ ℤ.

The nominal dynamics of system (4) are defined by
\[
z_{k+1} := Az_k + Bv_k
\]
(8)
where z_k ∈ ℜ^n is the nominal state and v_k ∈ ℜ^m is the
nominal control input.

Assumption 1: The system state x_k can be measured in real-
time and the pair (A, B) is stabilisable.

The goal of this paper is to design an MPC control law,
which steers the system trajectory from the state x_k at time
k to the equilibrium such that constraints (5) are satisfied.
For this, the control signal is specified as
\[
u_k = K(x_k - z_k) + v_k
\]
(9)
where K_k ∈ ℜ^n×m, for all k ∈ ℤ[1,∞).

Define the error \(e_k := x_k - z_k\) as the deviation of the
actual state from the nominal state. Under control (9), we obtain the
equation
\[
e_{k+1} = (A + BK_k)e_k + Bp_k
\]
(10a)
\[
q_k = (C_q + D_qK_k)e_k + C_qz_k + D_qv_k
\]
(10b)
p_k = Δz_kq_k
(10c)

Ellipsoids centred around the nominal trajectory z_k are de-
defined as
\[
P(z_k) := \{x \in ℜ^n | (x - z_k)^T P(x - z_k) \leq \frac{\alpha}{4}\}
\]
(11)
where positive-definite matrix P ∈ ℜ^n×n, and scalar \(\alpha > 0\)
are given.

By a modification of a result in [26], we first introduce a
preliminary lemma, called the ellipsoid mapping, which can
cast the actual system trajectory inside the ellipsoids centred
along the nominal trajectory.

Lemma 3 (Ellipsoid mapping): Consider system (4) and
the block diagonal perturbation constraints (7). Let z_k ∈ ℜ^n,
x_k ∈ P(z_k), P ∈ ℜ^n×n, is a positive-definite matrix and \(\alpha > 0\).
Suppose that there exist \(v_k ∈ ℜ^m, K_k ∈ ℜ^n×m, \xi_k ∈ [0, 1],\)
Λ_k = diag(ξ_{1k}I, . . . , ξ_{nk}I) with \(\lambda_{ik} > 0\) for all \(i ∈ ℤ[1,n]\), such that
\[
S_k := \begin{bmatrix}
-\xi_kP & * & * & * \\
0 & -Λ_k & * & * \\
0 & 0 & 4(\xi_k - 1) & * \\
A + BK_k & Bp_k & 0 & -P^{-1} \\
C_q + D_qK_k & 0 & C_qz_k + D_qv_k & 0 & -Λ_k
\end{bmatrix} \leq 0
\]
(12)

Then, the control law \(u_k = K_i(x_k - z_k) + v_k\) guarantees that
\(x_{k+1} ∈ P(z_{k+1})\), where \(z_{k+1} = Az_k + Bv_k\).

Proof: Under the control \(u_k = K_i(x_k - z_k) + v_k\), the require-
ment that \(x_{k+1} ∈ P(z_{k+1})\) is equivalent to the quadratic functional
\[
T_0 = \begin{bmatrix}
x_k - z_k \\
\end{bmatrix}^T \begin{bmatrix}
(A + BK_k) & Bp_k \\
0 & 0
\end{bmatrix} \begin{bmatrix}
(A + BK_k) & Bp_k \\
0 & 0
\end{bmatrix} \begin{bmatrix}
x_k - z_k \\
\end{bmatrix} - \frac{\alpha}{4} \leq 0
\]
(13)
The requirement that \(x_k ∈ P(z_k)\) is equivalent to
\[
T_1 = \begin{bmatrix}
x_k - z_k \\
0
\end{bmatrix}^T \begin{bmatrix}
P[I] & 0 \\
0 & 0
\end{bmatrix} \begin{bmatrix}
x_k - z_k \\
0
\end{bmatrix} - \frac{\alpha}{4} \leq 0
\]
(14)
Each of the m perturbation constraints (7) is equivalent to
\[
T_{2i} = \begin{bmatrix}
x_k - z_k \\
0
\end{bmatrix}^T \begin{bmatrix}
0 & I \\
I & 0
\end{bmatrix} \begin{bmatrix}
(C_q + D_qK_k) & 0 \\
0 & 0
\end{bmatrix} \begin{bmatrix}
x_k - z_k \\
0
\end{bmatrix} - 2(C_qz_k + D_qv_k)^T
\]
(15)
Via the S-Procedure, the requirement of the lemma is met if
there exists \(ξ_k \text{ and } Λ_k \text{ such that}
\[
T_0 - ξ_kT_1 - \sum_{i=1}^{m} λ_i^{-1} T_{2i} \leq 0
\]
(16)
The quadratic function (16) is a functional of \([x_k - z_k \ p_k]^T\),
which is required to hold for all \(x_k \text{ and } p_k\). With the
Schur complement and Lemma 2, and some simple matrix
transformations, (16) is reduced to the matrix constraints
\(S_i ≤ 0\).

Remark 2: Pre- and post-multiplying (12) by \([I \ 0 \ 0] \text{ and } [I \ 0 \ 0]^T\), we obtain
\[
\begin{bmatrix}
-ξ_kP & * \\
0 & -Λ_k
\end{bmatrix} \leq 0
\]
(17)
Using the Schur complement, this is equivalent to
\[
(A + BK_k)^T A + BK_k = ξ_kP \leq 0
\]
Since \(ξ_k \in [0, 1]\), this implies that the nominal system (8) is
asymptotically stable under the control law \(v_k := K_i(x_k - z_k)\).

Note that (12) represents an LMI since \(S_i \text{ is linear in its}
unknowns. Lemma 3 proposes a way to calculate a feedback
law \(K_i \text{ and an open-loop input } v_k \text{ such that } x_{k+1} ∈ P(z_{k+1})\)
if \(x_k ∈ P(z_k)\). The associated control input satisfies con-
straints (5) if the conditions stated in the following lemma are satisfied.

Lemma 4 (Input constraints) [26]: The control signal \(u_k\)
satisfies the input constraints (5) for all \(x_k ∈ P(z_k)\), if and only
if there exist \(v_k ∈ ℜ^m, K_k ∈ ℜ^n×m, \text{ and } ξ_{ik} > 0\), for

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all \( j \in \mathbb{Z}_{\{1,\ldots,n\}} \) such that
\[
U_{k,j} := \begin{bmatrix} \frac{-\eta_j P}{4} & * & * & * \\
\alpha g_j^T K_k & \alpha & 2g_j^T V_k - 2h_j \end{bmatrix} \leq 0, \quad \forall j \in \mathbb{Z}_{\{1,\ldots,n\}}
\] (17)

Remark 3: Note that Lemma 4 does not introduce any conservativeness because of its condition being ‘if and only if’.

The following lemma states conditions for the calculation of \( P \) and \( \alpha \), that were assumed to be given in the results above.

Lemma 5 (Parameters \( P \) and \( \alpha \)) [15]: Consider system (4) and the block diagonal perturbation constraints (7). Let \( Q \in \mathbb{R}^{n \times n} \) and \( R \in \mathbb{R}^{n \times n} \) be positive-definite matrices. If there exist \( \Lambda_a = \text{diag}(\lambda_{a1}, \ldots, \lambda_{am}) \) with \( \lambda_{ai} > 0 \), for all \( i \in \mathbb{Z}_{\{1,\ldots,m\}} \), positive-definite matrix \( X \in \mathbb{R}^{n \times n} \), non-square matrix \( Y \in \mathbb{R}^{n \times m} \), and \( \alpha > 0 \), such that
\[
\begin{bmatrix}
-X & * & * & * \\
R^T Y & -\alpha & * & * \\
Q X & 0 & 0 & -\Lambda_a & * \\
AX + BY & 0 & 0 & -X + B_p \Lambda_a B_p^T \\
\end{bmatrix} \leq 0
\] (18a)
\[
\begin{bmatrix}
h h^T & g^T Y \\
* & X \\
\end{bmatrix} \geq 0
\] (18b)
with \( j \in \mathbb{Z}_{\{1,\ldots,n\}} \), then, the ellipsoid
\[
\mathcal{X}_j := \{ x \in \mathbb{R}^n \mid x^T P x \leq \alpha \}
\] (19)
with \( P := \alpha X^{-1} \), and the linear state feedback control law
\[
u_k = F x_k \quad \text{with} \quad F := Y X^{-1}
\] have the following properties:

(1) Let \( M(x_k) := x_k^T P x_k \). Then, \( M(x_{k+1}) - M(x_k) \leq -x_k^T Q x_k - u_k^T R u_k \) for all \( k \in \mathbb{Z}_{\{0,\ldots,N\}} \) and for all \( x_k \in \mathcal{X}_j \);
(2) \( \mathcal{X}_j \) is robustly invariant for system (4) controlled by the feedback control law \( u_k = F x_k \);
(3) \( u_k = F x_k \in \mathcal{U} \) for all \( x_k \in \mathcal{X}_j \).

Define an ellipsoid \( \Omega_z \) as
\[
\Omega_z := \{ z \in \mathbb{R}^n \mid z^T P z \leq \alpha \}
\]
which will serve as the terminal set of the online optimisation problem in the following section.

If we choose \( v_k := F z_k \), and if the additional constraint stated in the next lemma is satisfied, the solution of (18a) also satisfies (12), and therefore, Lemma 3 is satisfied for all \( z_k \in \Omega_z \). This result plays an important role in the construction of a feasible solution to the proposed MPC scheme later.

Lemma 6: Let \( \mathcal{W} := (C_{\mathcal{F}} + D_{\mathcal{F}} F)^\top \alpha \Lambda_a^{-1} (C_{\mathcal{F}} + D_{\mathcal{F}} F) \), and let \( X, Y, \alpha \) and \( \Lambda_a \) be a feasible solution to (18a). Suppose there exists \( \xi \in [0,1) \) such that
\[
\begin{bmatrix}
(1 - \xi) P - \mathcal{W} & \mathcal{W} \\
\mathcal{W} & F^T R F + Q - (1 - \xi) P \\
\end{bmatrix} \geq 0
\] (20)
where \( P = \alpha X^{-1} \) and \( F = Y X^{-1} \). Then, \( X, Y, \alpha, \xi \) and \( \Lambda_a \) also satisfy (12) for all \( z_k \in \Omega \), with \( \Lambda_a := \alpha^{-1} \Lambda_a, K_k := F, \xi_k := \xi \) and \( v_k := F x_k \).

Proof: By the Schur complement, (18a) is equivalent to
\[
\begin{bmatrix}
F^T R F + Q - \mathcal{W} & \mathcal{W} \\
\mathcal{W} & -\alpha \Lambda_a^{-1} \\
\end{bmatrix} \geq 0
\] (21)
and (12) is equivalent to
\[
\begin{bmatrix}
-\xi P + \mathcal{W} & \mathcal{W} \\
\mathcal{W} & -\alpha \Lambda_a^{-1} \\
\end{bmatrix} \geq 0
\]
(22)
with \( v_k = F z_k \).

Obviously, \( \mathcal{W}^T = \mathcal{W} > 0 \). Define \( \beta := \frac{\alpha}{4}(\xi - 1) + z_k^T \mathcal{W} z_k \), and assume that \( \beta < 0 \), then (22) can be rewritten as
\[
\begin{bmatrix}
-\xi P + \mathcal{W} & \beta^{-1} z_k^T \mathcal{W} \\
\mathcal{W} & -\alpha \Lambda_a^{-1} \\
\end{bmatrix} \geq 0
\] (23)
Suppose that
\[
F^T R F + Q - \mathcal{W} + \mathcal{W} (F^T R F + Q - (1 - \xi) P)^{-1} \mathcal{W} z_k \leq \frac{\alpha}{4}(1 - \xi),
\]
\[\forall z_k \in \Omega_z \]
(24)
Then, satisfaction of (21) implies that (23) holds, that is, the solution of (14a) also satisfies (12). Since \( \beta < 0 \), using the Schur complement, (24) is equivalent to
\[
\begin{bmatrix}
F^T R F + Q - (1 - \xi) P & \mathcal{W} \\
\mathcal{W} & z_k^T \mathcal{W} z_k \end{bmatrix} \geq 0
\] (25)
which means that
\[
z_k^T (\mathcal{W} + \mathcal{W} (F^T R F + Q - (1 - \xi) P)^{-1} \mathcal{W}) z_k \leq \frac{\alpha}{4}(1 - \xi),
\]
\[\forall z_k \in \Omega_z \]
(26)
Note that \( \beta < 0 \) is satisfied automatically if inequality (25) has a feasible solution.

Since \( z_k \in \Omega_z \), that is, \( z_k^T P z_k \leq \frac{\alpha}{4} \), it follows that if we impose
\[
W + \mathcal{W} (F^T R F + Q + (\xi - 1) P)^{-1} \mathcal{W} \leq P(1 - \xi)
\] (27)
which is the Schur complement to (20), then (26), and thus, (24), holds. Therefore (21) implies (23), which completes the proof. \( \square \)

Unfortunately, (20) is a non-LMI, which is not easy to solve. Thus, normally we solve the LMIs (18a) and (b), and then check whether or not the parameters \( \Lambda_a, P, F \) satisfy (20).

Based on the foregoing lemmas, in the next section we propose a novel finite horizon MPC scheme and discuss its feasibility and stability properties if (20) is satisfied.

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3 Finite horizon MPC with ellipsoid mapping

In this section, we present the main result of this paper, namely a robust MPC scheme for the uncertain linear system (4). The control sequence obtained consists of two components: a nominal open-loop control sequence and a feedback control law. The idea is that the nominal control sequence steers the centre of predicted ellipsoids into a prescribed nominal terminal set, where the centres of the ellipsoids are generated by the nominal model (8). The feedback control law ensures that the state of the uncertain linear system lies in the ellipsoids for all admissible uncertainties. The open-loop component is similar to that of [24, 25, 30], but in our scheme the feedback control law is defined by an online determined time-varying matrix rather than a static one. As it will be shown, both the nominal control action and the feedback control law are obtained offline by repeatedly solving a convex optimisation problem. Suppose that $P$ and $\alpha$ are obtained offline by Lemma 5. To distinguish the actual state and predicted trajectories, in what follows the index $k + i/k$ denotes future values at time $k + i$ predicted at time $k$, $i \in \mathbb{Z}_{0\rightarrow N-1}$.

Denote the sequences $z_k := [z_k/k, z_{k+1/k}, \ldots, z_{k+N-1/k}]$ and $v_k := [v_{k/k}, v_{k+1/k}, \ldots, v_{k+N-1/k}]$, and define the nominal cost functional as

$$J(z_k, v_k) = \sum_{i=0}^{N-1} \left[ z_{k+i/k}^T Q z_{k+i/k} + v_{k+i/k}^T R v_{k+i/k} \right] + \sum_{i=0}^{N} z_{k+N/k}^T P z_{k+N/k} \quad (28)$$

where $N \in \mathbb{Z}_{0\rightarrow N}$ is the prediction horizon, $Q, R \in \mathbb{R}^{n \times n}$, and $P \in \mathbb{R}^{n \times n}$ are positive-definite weighting matrices, $z_{k+i/k}^T Q z_{k+i/k} + v_{k+i/k}^T R v_{k+i/k}$ is the stage cost and $z_{k+N/k}^T P z_{k+N/k}$ is the terminal penalty function.

The proposed MPC scheme is based on the repeated solution of the following optimisation problem for the current state $x_k \in \mathbb{R}^n$:

**Problem 1:**

minimise $J(z_k, v_k)$ \quad (29a)

subject to $z_{k+i/k} = A z_{k+i/k} + B v_{k+i/k}$, $z_{k/k} = z_k$ \quad (29b)

$x_k \in \mathbb{P}(z_k)$ \quad (29c)

$S_{k+i/k} \leq 0$, $i \in \mathbb{Z}_{0\rightarrow N-1}$ \quad (29d)

$U_{k+i/k} \leq 0$, $i \in \mathbb{Z}_{0\rightarrow N-1}$, $j \in \mathbb{Z}_{1\rightarrow r}$ \quad (29e)

$z_{k+N/k} \in \Omega$ \quad (29f)

where $\Lambda_{k+i/k} = \text{diag}(\lambda_{1,k+i/k}, \ldots, \lambda_{m,k+i/k})$, with $\lambda_{i,k+i/k} > 0$, $i \in \mathbb{Z}_{1\rightarrow r}$, $\eta_{i,k+i/k} > 0$ and $\xi_{i,k+i/k} \in (0,1)$. The index $k+i/k$ denotes the term $S_i$ in Lemma 3 and $U_{i,j}$ in Lemma 4, respectively, at the future time $k+i$ predicted at time $k$.

**Remark 4:** Since $P$ and $\alpha$ are determined a priori, Problem 1 is a convex optimisation problem [31]. Furthermore, the optimisation problem considered can be transformed into an SDP since (29c and f) can be rewritten as LMI's by the Schur complement. Thus, for simplicity, in what follows we refer to Problem 1 as an SDP.

**Remark 5:** In this paper, only input constraint (5) is considered. In order to take state constraint into account, similar to [32], a restricted constraint on the ellipse centres has to be estimated offline and involved in Problem 1.

We refer to the set $\Omega$, as the ‘nominal’ terminal region of Problem 1. The following lemma states that any feasible solution to Problem 1 steers the actual system into the set $\chi$ in $N$ steps. Since $\chi$ has been obtained using Lemma 5, we know that it is robustly invariant for the original system (4). Obviously, $\chi$ represents the ‘actual’ terminal region of the proposed scheme for system (4) with the static control law $u_k = F x_k$, which motivates us to use this control local as a candidate terminal control law, while $x_k \in \chi$.

**Lemma 7:** Let $z_k$ and $v_k$ denote a feasible solution to Problem 1. Then, the admissible state trajectory $x_k := \{x_k/k, x_{k+1/k}, \ldots, x_{k+N/k}\}$ satisfies $x_{k+N/k}^T P x_{k+N/k} \leq \alpha$, that is, $x_{k+N/k} \in \chi$:

Proof: Since $x_k \in \mathbb{P}(z_k)$, it follows by repeatedly exploiting Lemma 3 that $x_{k+N/k} - z_{k+N/k} \in \mathbb{P}(z_{k+N/k})$, that is

$$(x_{k+N/k} - z_{k+N/k})^T P (x_{k+N/k} - z_{k+N/k}) \leq \frac{\alpha}{4}.$$}

Note that for any $\beta_1, \beta_2 \in \mathbb{R}^n$ and positive-definite matrix $M \in \mathbb{R}^{n \times n}$, $\beta_1^T M \beta_1 + \beta_2^T M \beta_2 \geq \frac{1}{2} (\beta_1 + \beta_2)^T M (\beta_1 + \beta_2)$, which can be confirmed by a simple inner-product transformation. In virtue of this, we conclude that $x_{k+N/k}^T P x_{k+N/k} \leq \frac{\alpha}{4}$ and $x_{k+N/k} - z_{k+N/k))^T P x_{k+N/k} - z_{k+N/k) \leq \frac{\alpha}{2}$.

**Remark 6:** If $b > 0$ and $c > 0$ such that $\frac{1}{b} + \frac{1}{c} = \frac{1}{2}$. Define the sets $\mathbb{P}_b(z_k) := \{x \mid (x_k - z_k) \in \mathbb{P}(x_k - z_k) \leq \frac{1}{b}\}$ and $\Omega_b := \{z_k \mid z_k^T P z_k \leq \frac{2}{b}\}$. If Problem 1 has a feasible solution, then, by similar arguments as in the proof of Lemma 7 we can guarantee that $x_{k+N/k}^T P x_{k+N/k} - \alpha$ holds. In other words, $\mathbb{P}_b(z_k)$ and $\Omega_b$ can serve as the mapping ellipsoid and the ‘nominal’ terminal set, respectively, of Problem 1.

We are now ready to present the robust MPC scheme that is based on the ellipsoid mapping, and establish its feasibility and robust stability properties.

First, we assume that the following assumption is satisfied.

**Assumption 2:** Let the pair $(P, \alpha)$ be a feasible solution to (18), and suppose that this pair also satisfies (20).

The robust MPC control law is derived from the solution of the convex optimisation Problem 1, which is solved repeatedly at each sampling instant $k$ based on the state measurement $x_k$.

**Algorithm 1:**

Step 1. At time instant $k$, measure the state $x_k$ and solve Problem 1.

Step 2. Apply $u_k = v_k + \Lambda_k x_k$ to the actual system (4). Set $k = k + 1$ and go to Step 1.

The feasibility and stability properties of the proposed scheme according to Algorithm 1 depend on the feasibility of Problem 1 at the initial time instant.
Let $z^*_i$ and $v^*_{k+i/k}$, $K^*_{k+i/k}$, $b^*_{k+i/k}$, $k^*_{k+i/k}, \Lambda^*_{k+i/k}$, $n_{k+i/k}$, $i \in \mathbb{Z}_{0[N-1]}$ be the optimal solution to Problem 1. For brevity of notation, denote $\Gamma_{k+i/k} := [v^*_{k+i/k}, K^*_{k+i/k}, b^*_{k+i/k}, k^*_{k+i/k}, \Lambda^*_{k+i/k}, n_{k+i/k}]$ for all $i \in \mathbb{Z}_{0[N-1]}$.

**Theorem 1 (Feasibility):** Problem 1 is feasible at all time instants if it is feasible at $k = 0$.

**Proof:** Assume that Problem 1 is feasible at time instant $k$, and its solution is

$$[z^*_{k+1/k}, \Gamma^*_{k+1/k}, \Gamma^*_{k+1/k}, \ldots, \Gamma^*_{k+N-1/k}]$$

(30)

Denote the actual state sequence and the nominal state sequence, related to (30), as $[x_{k+1/k}, x_{k+2/k}, \ldots, x_{k+N/k}]$ and $[z^*_{k+1/k}, z^*_{k+2/k}, \ldots, z^*_{k+N/k}]$. According to Algorithm 1, the input applied to the system (4) at time $k$ is $u_k = K^*_{k+i/k}(x_k - z^*_i) + v^*_{k+i/k}$. By Lemma 3, the input guarantees that the actual system state $x_{k+1}$ lies in the ellipsoid $P(z^*_{k+1/k})$ for any admissible uncertainty.

Denote $\Gamma_{k+i/k} := [F^*_{k+i/k}, F^*_{k+i/k}, \xi_k, \eta_k]$. With state $x_{k+1}$ at time instant $k + 1$, consider the following feasible solution candidate

$$[z^*_{k+1/k}, \Gamma^*_{k+1/k}, \Gamma^*_{k+1/k}, \ldots, \Gamma^*_{k+N-1/k}, \Gamma_{k+N/k+1}]$$

(31)

where $F$ and $A$ satisfy (18), and $\xi$ and $\eta$ will be introduced later.

Based on Lemma 7 and $z^*_{k+N/k} \in \Omega_z$, we know that $x_{k+N/k} \in \Omega_x$ and $F_{x_{k+N/k}}$ satisfies the input constraints (5), that is, $g^T_k F_{x_{k+N/k}} \leq h_j$. Obviously, $F_{x_{k+N/k}} = F_{z_{k+N/k}} + F(x_{k+N/k} - z^*_{k+N/k})$. From the proof of Lemma 4, it follows that if there exists $\eta > 0$ such that

$$2g^T_k F(x_{k+N/k} - z^*_{k+N/k}) + 2g^T_k v_{k+N/k} - 2h_j \eta$$

then, inequality (17) is satisfied. In terms of $v_{k+N/k+1} := F_{z_{k+N/k}} + \eta \in \Omega_v$, this is equivalent to

$$2g^T_k F_{x_{k+N/k}} - 2h_j \eta$$

then, inequality (17) is satisfied. In terms of $v_{k+N/k+1} := F_{z_{k+N/k}} + \eta \in \Omega_v$, this is equivalent to

$$2g^T_k F_{x_{k+N/k}} - 2h_j \eta$$

(32)

Since $g^T_k F_{x_{k+N/k}} \leq h_j$, and $(x_{k+N/k} - z^*_{k+N/k})^T P(x_{k+N/k} - z^*_i) \leq \frac{\gamma}{3}$, which results from Lemma 5, obviously, there exists a scalar $\eta > 0$ satisfying condition (32). Therefore at time instant $k + 1$, constraint (29e) is satisfied with $v_{k+N/k+1} \eta = 0$ and $K_{k+N/k} \eta := F$.

Based on Assumption 2 and Lemma 6, there exists $\xi$ such that constraint (29d) is satisfied with $\xi_k v_{k+N/k+1}$, $K_{k+N/k+1}$ and $A_{k+N/k+1} := A - BF_{z_{k+N/k}}$ satisfies (29f).

Based on the above discussion, sequence (31) is a feasible solution to Problem 1 at time instant $k + 1$.

Let us define a Lyapunov function candidate as

$$V(x_k) := \min_{z_{k+N/k+1}, K_{k+N/k+1}, A_{k+N/k+1}, n_{k+N/k+1}} J(z_k, v_k)$$

(33)

Here, we emphasise that the optimal value of the cost functional is solely defined by the state $x_k$, which is measured online.

**Theorem 2 (Stability):** Suppose that Problem 1 is feasible at time $k = 0$. Then, system (4) is asymptotically stabilised under the proposed MPC control law according to Algorithm 1.

**Proof:**

1. $0 < V(x_k) < +\infty$ for all $x_k \neq 0$, which follows directly from the definition of $V(x_k)$.
2. $V(0) = 0$, which is confirmed by choosing all the terms of sequences $x_k$ and $v_k$ as 0.
3. Assume that Problem 1 is feasible at time instant $k$, and its solution is given by (30). Then

$$V(x_k) = \sum_{i=0}^{N-1} z^*_{k+i/k} Qz^*_{k+i/k} + v^T_{k+i/k} Rv^*_{k+i/k} + z^T_{k+N/k} Pz^*_{k+N/k}$$

(34)

Denote $z_{k+1} := [z_{k+1/k}, z_{k+2/k}, \ldots, z_{k+N/k}]$ and $v_{k+1} := [v_{k+1/k}, v_{k+2/k}, \ldots, v_{k+N/k}]$. Since (31) is a feasible solution to Problem 1 at time instant $k + 1$, we have

$$J(z_{k+1}, v_{k+1}) = \sum_{i=0}^{N-1} z_{k+i+1/k}^T Qz_{k+i+1/k} + v_{k+i+1/k}^T Rv_{k+i+1/k} + z_{k+N/k}^T Pz_{k+N/k}$$

(35)

Owing to the Principle of Optimality, we have $V(x_{k+1}) \leq J(z_{k+1}, v_{k+1})$. Thus

$$V(x_{k+1}) - V(x_k) = J(z_{k+1}, v_{k+1}) - V(x_k) = z^T_{k+N/k} Qz^*_{k+N/k} + z^T_{k+N/k+1} Pz^*_{k+N/k+1} + z^T_{k+N/k+1} Rv^*_{k+N/k+1} - z_{k+N/k}^T Pz_{k+N/k}$$

(36)

Since $z^T_{k+N/k+1} Pz_{k+N/k+1} \leq z^T_{k+N/k} Pz_{k+N/k}$, which results from Lemma 5, $V(x_{k+1}) \leq V(x_k)$.

Clearly, $V(x_k)$ is a Lyapunov function and thus, system (4) is asymptotically stabilised [2] by the control (9).$

**Remark 7:** Problem 1 is based on the prediction of the future nominal trajectory associated with the nominal system (8). Although an exact prediction of the actual trajectory is not possible in the presence of uncertainties, we know that the actual system trajectory lies in the prescribed ellipsoids centred around the nominal trajectory with respect to any admissible uncertainty.

Assumption 2 is strong and plays an important role in the construction of feasible solution to Problem 1, as well as in the proof of stability of the proposed finite horizon MPC scheme.
4 Numerical example

Consider the discrete-time linear system
\[ x_{1,k+1} = x_{1,k} + (0.15 + 0.125 \rho_k) x_{2,k} \]
\[ x_{2,k+1} = 0.15 x_{1,k} + (0.5 - 0.125 \rho_k) x_{2,k} + 0.1 \kappa u_k \]
subject to the input constraint \( [0.5 \quad -0.5]^T u_k \leq [1 \quad 1]^T \), where \( \kappa = 0.8 \) and \( x_{j,k}, j = 1, 2 \), is the \( j \)th element of vector \( x_k \). The time-varying parameter \( \rho_k \) is bounded by \( \rho_k \in [-1 \quad 1] \), for all \( k \in \mathbb{Z}_{[0, \infty)} \). Thus, in the notations of (4), \( A = \begin{bmatrix} 1 & 0.15 \\ 0.15 & 0.5 \end{bmatrix}, B = [0 \quad 0.1 \kappa]^T, B_p = [0.25 \quad -0.25]^T, C_q = [0 \quad 0.5] \) and \( D_{qu} = 0 \). The uncertainty is described by...
Exemplary time profiles for dynamic responses and input trajectory with Algorithm 1 from the initial state $x_0 = [0.55, 0.60]^T$. We choose the matrices $Q = \text{diag}(1, 1)$ and $R = 1$ in the cost functional (28).

The matrix $P = \begin{bmatrix} 1059.7 & 219.8 \\ 219.8 & 171.3 \end{bmatrix}$ and the scalar $\alpha = 50$ are obtained by solving (18). Furthermore, $P$ and $\alpha$ do satisfy Assumption 2. Thus, we use the finite horizon MPC with ellipsoid mapping according to Algorithm 1.

Exemplary, Fig. 2 shows simulation results for the initial state $x_0 = [0.25, 0.30]^T$ corresponding to $\rho_k = 0.5$, for all $k \in \mathbb{Z}_{[0,\infty)}$. The solid line shows the state and input trajectories obtained by the proposed method with prediction horizon $N = 10$. The dashed line shows the trajectories obtained by [15]. The performance of the proposed finite
horizon MPC with ellipsoid mapping is worse than the one of [15] since only nominal performance rather than robust performance [15] is minimised in Problem 1. However, for $x_0 = [0.55 \ 0.60]^T$ and $x_0 = [0.65 \ 0.75]^T$, the closed-loop MPC scheme [15] has no feasible solution at the initial time instant, whereas the proposed finite horizon MPC scheme guarantees recursive feasibility and stability with the prediction horizons $N = 38$ and $N = 48$, respectively. This shows that the proposed finite horizon MPC scheme has a different region of attraction compared with [15]. Since part of the admissible input is used to keep the actual state in the ellipsoids around the nominal trajectory, a large prediction horizon is required in the proposed finite horizon MPC with ellipsoid mapping. Fig. 3 shows the simulation results for the initial state $x_0 = [0.55 \ 0.60]^T$, whereas $\rho_0 = 0.5$. The simulation shows that stability as well as constraint satisfaction are guaranteed even if the initial state is far from the equilibrium.

5 Conclusions

In this paper, we proposed an MPC scheme for discrete-time linear systems with structured feedback uncertainty and constraints. The control signal is constructed by both feedback and open-loop terms, which are calculated online by solving a convex optimisation problem. The open-loop component steers the centre of associated ellipsoids into a terminal set, while the feedback component keeps the system state in those ellipsoids for all admissible uncertainties. If the optimisation problem is initially feasible, the proposed MPC strategy guarantees recursive feasibility and closed-loop stability. A simulation example illustrated the effectiveness of the derived theory.

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7 References