Terminal Set of Min-max Model Predictive Control with Guaranteed $L_2$ Performance

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Abstract—We present an approach to synthesize the terminal control law and the terminal set of nonlinear model predictive control with guaranteed $L_2$ performance, where a structured feedback uncertainty is considered. The terminal control law, which is obtained through the solution to a Hamilton-Jacobi-Bellman-Isaacs (HJBI) inequality of the Jacobian linearization of the original nonlinear system, renders the uncertain nonlinear system robust invariant in the terminal set.

I. INTRODUCTION

In order to achieve robustness, a control law must stabilize the considered system for all possible realizations of the uncertainty. In model predictive control (MPC), an intuitive approach is to solve a min-max optimization problem online in the presence of disturbances and/or model mismatches [1–3]. The optimization problem can be solved either in open-loop or in closed-loop fashion. The trajectory of the open-loop scheme [1] may diverge from the trajectory of its prediction. This causes the feasible region of the optimization problem to shrink or even become empty for a reasonable prediction horizon, as only the open-loop control is used during the sampling time and the disturbances are not directly rejected between the sampling instances. Therefore, the optimization problem of min-max MPC should search for a feedback strategy rather than an open-loop control sequence [3]. This has already been accepted as one of the keystones [4,5] of robust MPC. Choosing the stage cost as $z^T \cdot - \gamma w^T w$ for a fixed constant $\gamma > 0$ and optimizing over the finite horizon control action or control law, min-max MPC schemes with guaranteed $L_2$ performance are proposed in [1,3,6,7], where $z$ is the control output of the system and $w$ the exogenous input. The input-to-state stability (ISS) of min-max MPC is discussed in [8–10]. Note that for linear systems, $L_2$ performance is also called $H_\infty$ performance, and $L_2$ control problem is called $H_\infty$ control problem.

Min-max MPC schemes [1,3,7] depend on the offline computation of a suitable terminal penalty, a terminal cost and a terminal control law, where the system under the terminal control law is robust invariant with respect to the disturbances or model-plant mismatches in the terminal set, and the terminal set is a sub-level set of the terminal penalty [8,10,11]. Although min-max MPC is widely discussed, there is only a few results on how to choose the terminal set and the terminal penalty. A scheme based on the solution to a partial differential inequality, which is in general a hard problem, is proposed in [1]. There a nonlinear terminal control law is considered. In order to avoid the difficulty, a linear feedback matrix based on the Jacobian linearization of the considered nonlinear systems is calculated in [3]. The terminal control law is a nonlinear control law, which is consisted of the linear feedback matrix and a nonlinear state-dependent perturbation term. The paper [12] presents a linear matrix inequality (LMI) based method to solve a terminal set, where a linear terminal control is considered.

In this paper, a linear terminal control law is solved for the min-max MPC with guaranteed $L_2$ performance, where a nonlinear system with respect to structured feedback uncertainties is considered. The proposed scheme is based on the Jacobian linearization of the considered nonlinear systems, where an algebraic Riccati inequality rather than a partial differential inequality is solved offline. The robust invariance property of the considered nonlinear system under the linear control law in the terminal set is discussed in detail. Compared with the existing results, the existence of the terminal set is proved. As an alternative, an LMI based scheme to solve the terminal control law and the terminal set instantaneously is proposed. We also show that min-max MPC with guaranteed $L_2$ performance can be used to deal with unknown but bounded external disturbances, and give the estimation on the upper bound of the disturbances that min-max MPC scheme can deal with.

The paper is structured as follows. In Section II we state the problem setup and the main results of min-max MPC with guaranteed $L_2$ performance. A scheme for choosing the terminal set is proposed in Section III, which is based on the solution to an algebraic inequality. In Section IV, the possibility that min-max MPC is used to deal with the control of nonlinear systems with unknown but bounded disturbances is discussed.

A. Notations and Basic Definitions

Let $\mathbb{R}$ denote the field of real numbers, $\mathbb{R}^n$ the $n$-dimensional Euclidean space. For a vector $v \in \mathbb{R}^n$, $\|v\|$ the 2-norm. For a signal $v(\cdot) : \mathbb{R} \rightarrow \mathbb{R}^n$, $\|v\|_{L_2}$ denotes the time-domain 2-norm, that is, $\|v\|_{L_2} = \int_0^\infty v(\tau)^T v(\tau) \, d\tau < \infty$. Suppose that $M \in \mathbb{R}^{n \times n}$, $\lambda_{\min}(M)$ ($\lambda_{\max}(M)$) is the smallest (largest) real part of eigenvalues of the matrix $M$. Moreover, $*$ is used to denote the symmetric part of a matrix.
i.e., \[ \begin{bmatrix} a \\ b \\ c \end{bmatrix} T \begin{bmatrix} a \\ b \\ c \end{bmatrix} \] = \[ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \].

The term \( \text{Co}\{\cdot\} \) denotes the convex hull of a set.

II. Problem Setup and Preliminary Results

Consider continuous-time nonlinear systems
\[
\dot{x} = f(x, u) + g(x, u)w,
\]
\[
z = h(x, u),
\]
where \( x(t) \in \mathbb{R}^{n_x} \) is the state, \( u(t) \in \mathbb{R}^{n_u} \) is the control input, \( w(t) \in \mathbb{R}^{n_w} \) is the disturbance, and \( z(t) \in \mathbb{R}^{n_z} \) is the output to be controlled. The system is subject to state and input constraints
\[
x(t) \in \mathcal{X}, \quad u(t) \in \mathcal{U}, \quad \forall t > 0.
\]

Fundamental assumptions for (1) and (2) are as follows:

**Assumption 1:** \( f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \rightarrow \mathbb{R}^{n_x}, \ g : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \rightarrow \mathbb{R}^{n_z}\) are twice continuously differentiable and \( f(0, 0) = 0 \) and \( h(0, 0) = 0 \). Thus, \( x = 0 \) is an equilibrium of the system for input \( w = 0 \).

**Assumption 2:** \( \mathcal{U} \subset \mathbb{R}^{n_u} \) is compact, \( \mathcal{X} \subset \mathbb{R}^{n_x} \) is connected and the point \((0, 0)\) is contained in the interior of \( \mathcal{X} \times \mathcal{U} \).

**Assumption 3:** The system (1) is locally zero-state detectable in a set \( \Theta \) which contains the origin as an interior, that is, for all \( x \in \Theta \subset \mathcal{X} \),
\[
\{z(t) \equiv 0, \forall t \geq 0\} \Rightarrow \{\lim_{t \rightarrow \infty} x(t) = 0\}.
\]

We confine ourselves to the case where all states are assumed to be measurable instantaneously. Furthermore, we assume that the uncertainty is a structured feedback uncertainty, that is,
\[
w = \Delta(x, u)z,
\]
where \( \Delta(x, u) \in \mathbb{R}^{n_x \times n_z} \) is an arbitrary smooth matrix valued function satisfying
\[
\|\Delta(x, u)\| \leq 1.
\]

That is, \( w \in \mathcal{W} \) where
\[
\mathcal{W} := \{w \in \mathbb{R}^{n_w} | w = \Delta(x, u)z, \|\Delta(x, u)\| \leq 1\}.
\]

The \( L_2 \) gain of the system (1) from \( w \) to \( z \) is defined as [13]
\[
\gamma := \sup \{\|z\|_{L_2} \|w\|_{L_2} \leq 1\}.
\]

Min-max MPC with guaranteed \( L_2 \) performance is designed to model predictive control law such that the closed-loop system is internally stable with respect to the constraints and the considered uncertainty, and \( L_2 \) performance of \( \gamma \leq \gamma_0 \), where \( \gamma_0 > 0 \) is a constant.

For any pairs \((u, w)\) on \([t, t + T_p]\), the finite horizon cost functional of min-max MPC is defined by
\[
J(\kappa(x), w, x(t)) := \mathcal{E}(x(t + T_p)) + \frac{1}{2} \int_t^{t + T_p} \left( z(\tau)^Tz(\tau) - \gamma_0^2w(\tau)^Tw(\tau) \right) d\tau,
\]
where the initial state of the trajectories is denoted by \( x(t), T_p \) is the prediction horizon and \( \mathcal{E}(x(t + T_p)) \) is a terminal penalty term. The finite horizon cost functional is to be minimized by the control strategy \( u := \kappa(x, t) \) with \( \kappa : \mathbb{R}^{n_x} \times \mathbb{R} \rightarrow \mathbb{R}^{n_u} \) for the worst case disturbance inputs \( w \). Note that a feedback control law is adopted in the optimization problem which is solved online, since a feedback control is superior to an open-loop control action while uncertainty is present [3, 5].

Since the terminal control law and the associated terminal control set of min-max MPC play an important role in the proof of robust analysis and the design of MPC control law, in this paper, we will mainly discuss how to choose them. Before it, we give an overview of the main results of min-max MPC, see [1, 3] in detail.

A. Min-max Model Predictive Control

Denote
\[
\mathcal{X}_f := \{x \in \mathbb{R}^{n_x} | \mathcal{E}(x) \leq \frac{\alpha}{2}\},
\]
where \( \mathcal{E} : \mathbb{R}^{n_x} \rightarrow [0, \infty) \) is a continuous-differentiable positive definite function and \( \alpha > 0 \). The set \( \mathcal{X}_f \) is a level set of function \( \mathcal{E}(x) \) and \( 0 \in \mathcal{X}_f \).

Min-max MPC solves online the following optimization problem.

**Problem 1:**

minimize \( \kappa(\cdot, \cdot, \cdot) \)

maximize \( w(\cdot, \cdot, \cdot) \)

subject to
\[
\dot{x} = f(x, \kappa(x, t)) + g(x, \kappa(x, t))w,
\]
\[
z = h(x, \kappa(x, t)), \quad x(t; x(t), t) = x(t),
\]
\[
x(\tau; x(t), t) \in \mathcal{X}, \quad \tau \in [t, t + T_p],
\]
\[
\kappa(x(t; x(t), t), \tau) \in \mathcal{U}, \quad \tau \in [t, t + T_p],
\]
\[
x(t + T_p; x(t), t) \in \mathcal{X}_f,
\]
\[
w \in \mathcal{W},
\]
where \( J(\cdot, \cdot, \cdot) \) is defined in Equation (6), \( x(\cdot; x(t), t) \) denotes the state trajectory starting from \( x(t) \) at time instant \( t \) under the control \( \kappa \) and \( \kappa \) denotes the related input function.

Assuming that the maximum and the minimum are attained, the optimal solution to Problem 1 is given by the optimal input trajectories, and
\[
w^*(\tau; x(t), t) := \arg \max_{w(\cdot)} J(\kappa^*(x, t), w, x(t)),
\]
\[
\kappa^*(x(t; x(t), t), \tau) := \arg \min_{\kappa(\cdot, \cdot, \cdot)} J(\kappa(x, t), w^*, x(t)),
\]
for all \( \tau \in [t, t + T_p] \).

**Remark 2.1:** Applying a feasible solution to the optimization problem at the initial time instant is sufficient to guarantee recursive feasibility, robust stability and performance, which is one of the main advantages of the MPC scheme which has a terminal penalty and a terminal set.

**Definition 1:** \( \pi : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_u}, \mathcal{X}_f \) and \( \mathcal{E}(x) \) are said to be the terminal control law, the terminal set and the terminal
penalty function, respectively, suppose that the following conditions are satisfied,

B0. $X_f \subseteq X$,
B1. $\pi(0) = 0$, and $\pi(x) \in \mathcal{U}$ for all $x \in X_f$,
B2. $E(0) = 0$, and $E(x)$ satisfies
\[ \frac{\partial E}{\partial x} f(x, \pi(x)) + \frac{1}{2} h^T(x, \pi(x)) h(x, \pi(x)) \\
+ \frac{1}{2} \frac{\partial E}{\partial x} g(x, \pi(x)) g^T(x, \pi(x)) \left( \frac{\partial E}{\partial x} \right)^T \leq 0, \tag{9} \]
for all $x \in X_f$.

**Proposition 1:** The terminal set $X_f$ has the following additional properties:

1. The terminal set $X_f$ is closed and connected due to the continuity of $E(x)$.
2. In the terminal set $X_f$, the system (1) controlled with $\pi(x)$ is finite horizon $L_2$ stable and its $L_2$ gain is less than or equal to $\gamma_0$ [14].
3. Since inequality (9) is sufficient to
\[ E(x(\tau)) - E(x(0)) \leq \frac{1}{2} \int_0^\tau (\|w\|^2 - \|z\|^2) \, dt, \]
and $w = \Delta(x, u)z$ and $\|\Delta(x, u)\| \leq 1$, we have
\[ E(x(\tau)) - E(x(0)) \leq 0 \]
for all $\tau \geq 0$. That is, $X_f$ is robust invariant for the nonlinear system (1) under the control law $u = \pi(x)$.

The MPC control law is defined by the following algorithm.

**Algorithm 1 Min-max MPC**

1. At sampling time $t$, measure the state $x(t)$ and solve Problem 1, and get the optimal solution \((w^*, (\tau, x(t), t), \kappa^*(x(t), x(t), t), \tau))\), for all $\tau \in [t, t + T_p]$.
2. Apply the control input
\[ \kappa^*(\cdot, \tau) := \kappa^*(x(t); x(t), t, \tau), \forall \tau \in [t, t + \delta], \tag{10} \]
to the system (1), where $\delta$ is the sampling time that Problem 1 is solved in discrete time.
3. Set $t := t + \delta$ and go to Step 1.

Now we are ready to collect the main results of min-max MPC with guaranteed $L_2$ performance, and the corresponding proof can be found in [1, 3, 4].

**Theorem 1:** Suppose that

(a) Assumptions 1-3 are satisfied,
(b) for the nonlinear system (1), there exist a locally asymptotically stabilizing controller $u = \pi(x)$, a terminal region $X_f$ defined by (7), and a continuously differentiable, positive definite function $E(x)$ that satisfies (9) for all $x \in X_f$,
(c) Problem 1 is feasible at time $t = 0$.

Then,

1. Problem 1 is feasible for all $t \geq 0$,
2. the constraints (2) are satisfied,
3. the system under control is internally stable, for all admissible disturbances $w$,
4. $L_2$ performance of $\gamma \leq \gamma_0$ is guaranteed.

**Proof:** See [1] for the proof.

On the one hand, only finite horizon optimization problem is considered in most of MPC schemes; On the other hand, stability or robust stability is an infinite time property of the systems. The constraint that the terminal state enters into the terminal set, which is an invariant set or robust invariant set of the considered system under the terminal control law, provides an effective way to deal with the dilemma. Thus, invariant sets or robust invariant sets play an important role in the analysis and synthesis of MPC.

### III. Terminal Control Law and Terminal Set

In this section, the existence of the terminal control law and the terminal set is discussed. Suppose that there is a linear control law such that the Jacobian linearization system of the original nonlinear system has the given attenuating rate of $\gamma_0$. It is shown that the linear control law $u = Kx$ can serve as the terminal control law. In accordance with it, an ellipsoidal terminal set and a quadratic terminal penalty can be determined. As an alternative, a scheme based on linear matrix inequality (LMI) is proposed if symmetrically polyhedral state and input constraints are considered.

We consider the Jacobian linearization of the system (1) at the origin
\[ \dot{x} = Ax + Bu + Gw, \]
\[ z = Cx + Du, \tag{11} \]
where $A = \frac{\partial f}{\partial x}(0, 0)$, $B = \frac{\partial f}{\partial u}(0, 0)$, $C = \frac{\partial h}{\partial x}(0, 0)$, $D = \frac{\partial h}{\partial u}(0, 0)$ and $G = g(0, 0)$.

**Theorem 2:** Let $\tau > 0$ be a given constant and suppose there exist $P \in \mathbb{R}^{n_x \times n_x}$ with $P > 0$ and $K \in \mathbb{R}^{n_u \times n_x}$ such that
\[ (A + BK)^T P + P(A + BK) + \frac{1}{\gamma_0^2} PGG^T P \]
\[ + 2(C + DK)^T (C + DK) \leq -\tau I. \tag{12} \]
Then, there exists a constant $\alpha \in (0, \infty)$ specifying a neighborhood $X_f$ of the origin in the form of
\[ X_f := \{x \in \mathbb{R}^{n_x} \mid x^T P x \leq \alpha\}, \tag{13} \]
such that the linear control law $Kx$, the term $E(x) := \frac{1}{2} x^T P x$ and the ellipsoid $X_f$ can serve as the terminal control law, the terminal penalty and the terminal set, respectively.

**Proof:** Since (12) is locally sufficient to (9), the Jacobian linearized system (11) is finite-gain $L_2$ stable and its $L_2$ gain is less than or equal to $\gamma_0$ [14].

Denote a set $X_0 := \{x \in \mathbb{R}^{n_x} \mid x^T P x \leq \alpha_1\}$ with $\alpha_1 > 0$ such that $X_0 \subseteq X$ and $Kx \in \mathcal{U}$ for all $x \in X_0$. That is, the input and state constraints (2) are satisfied in the set $X_0$. Denote $H(x, u, w) := \frac{1}{2} E(x) + \frac{1}{2} (\|w\|^2 - \gamma_0^2 \|w\|^2)$, $d(x, w) := f(x, Kx) - (A + BK)x + [g(x, Kx) - G]w$, and $s(x) := h(x, Kx) - (C + DK)x$, where $d(x, w)$ and $s(x)$ is...
the deviation of the original nonlinear system (1) and the linearized system (11) under the terminal control law. Thus, the nonlinear system (1) under the linear control law $Kx$ can be written as

$$
\dot{x} = (A + BK)x + Gw + d(x, w), \\
z = (C + DK)x + s(x) \\
w = \Delta(x, K)x. 
$$  \(14\)

In the set $\mathcal{X}_0$, $\|h(x, Kx)\|$ is bounded, since the set is a compact set and $h(\cdot, \cdot)$ is twice continuously differentiable. Furthermore, in the set $\mathcal{X}_0$,

$$
\|d(x, w)\| \leq \|f(x, Kx) - (A + BK)x\| + \|g(x, Kx) - G\| \cdot \|h(x, Kx)\|
$$

and

$$
\|s(x)\| \leq \|h(x, Kx) - (C + DK)x\|.
$$

The difference between the nonlinear systems (1) and its Jacobian linearization is higher-order infinitesimal of $x$ since only structured feedback uncertainty (3) is considered, and $s(0) = 0$ and $d(0, 0) = 0$.

Therefore, there exist positive scalars $L_d$ and $L_s$ such that

$$
\begin{align*}
\|d(x, w)\| &\leq L_d\|x\|, \\
\|s(x)\| &\leq L_s\|x\|,
\end{align*}
$$  \(15\)

where $L_d := \sup \left\{ \|d(x, w)\| : x \in \mathcal{X}_0, x \neq 0 \right\}$ and $L_s := \sup \left\{ \|s(x)\| : x \in \mathcal{X}_0, x \neq 0 \right\}$. Note that $L_d$ and $L_s$ exist, and the value of $L_d$ and $L_s$ depend on the size of the set $\mathcal{X}_0$. Furthermore, $\lim_{\alpha_1 \to 0} L_d = 0$ and $\lim_{\alpha_1 \to 0} L_s = 0$.

Denote

$$
H(x, u, w) := \frac{1}{2}(\|z\|^2 - \|w\|^2) + E(x).
$$

Considering the system dynamic of the original nonlinear system under the linear control law $u = Kx$, we have

$$
H(x, Kx, w) = \frac{1}{2}(\|h(x, Kx)\|^2 - \gamma_0^2\|w\|^2) + \frac{1}{2} \left( f(x, Kx) + g(x, Kx)w \right)^T P + x^T P \left( f(x, Kx) + g(x, Kx)w \right).
$$

That is,

$$
H(x, Kx, w) = \frac{1}{2} \left( 2x^T (C + DK)^T s(x) - \gamma_0^2\|w\|^2 \right) + s^T(x)s(x) + x^T (C + DK)^T (C + DK)x \\
+ \frac{1}{2} \left( (A + BK)x + Gw + d(x, w) \right)^T P + x^T P \left( (A + BK)x + Gw + d(x, w) \right).
$$  \(16\)

Considering the inequality (12), the inequality (16) can be rewritten as

$$
H(x, Kx, w) \leq -\frac{\tau}{2} x^T x + x^T Pd(x, w) + x^T (C + DK)^T s(x) + s^T(x)s(x).
$$  \(17\)

Due to the estimation of $s(x, w)$ and $d(x)$ in (15), we have

$$
H(x, Kx, w) \leq -\frac{\tau}{2} x^T x + \|P\| \cdot L_d \cdot \|x\|^2 + \|C + DK\| \cdot L_s \cdot \|x\|^2 + L_s^2 \cdot \|x\|^2 \\
\leq \left( -\frac{\tau}{2} + \|P\| \cdot L_d + \|C + DK\| \cdot L_s + L_s^2 \right) \cdot \|x\|^2.
$$

Now we choose an $\alpha \in (0, \alpha_1]$ such that in $\mathcal{X}_f$,

$$
\|P\| \cdot L_d + \|C + DK\| \cdot L_s + L_s^2 \leq \frac{\tau}{2}.
$$  \(18\)

Then, $H \leq 0$ in the set $\mathcal{X}_f$. Therefore, $\mathcal{X}_f$ is a terminal set and $Kx$ is a terminal control law since the inequality (9) and the constraints (2) are satisfied in the set $\mathcal{X}_f$ for the original nonlinear system (1) under the linear control law $Kx$.

From the above discussion, Algorithm 2 is proposed to determine a terminal control law $Kx$ and a terminal region $\mathcal{X}_f$ offline such that inequality (9) holds true and the constraints (2) are satisfied.

Algorithm 2 Terminal set and terminal penalty

1: Solve the standard linear $H_\infty$ control problem based on the Jacobian linearization of (11) to get a linear feedback gain $K$.

2: Solve the inequality (12) to get a positive definite matrix $P$ and a scalar $\tau > 0$.

3: Find $\alpha_1 > 0$ such that $\mathcal{X}_1 \subseteq \mathcal{X}$ and $Kx \in U$ for all $x \in \mathcal{X}_1$.

4: Find $\alpha \in (0, \alpha_1]$ such that inequality (18) is satisfied in $\mathcal{X}_f$.

A. LMI-based solution

Suppose that system (1) is required to satisfy the symmetrically polyhedral state and input constraints $\begin{bmatrix} x \\ u \end{bmatrix} \in \Xi$, where

$$
\Xi := \left\{ \begin{bmatrix} x \\ u \end{bmatrix} \in \mathbb{R}^{n_x+n_u} : c_j x + d_j u \leq 1, \forall j \in \mathbb{Z}_{[1,p]} \right\},
$$  \(19\)

and $c_j \in \mathbb{R}^{1 \times n_x}$ and $d_j \in \mathbb{R}^{1 \times n_u}$, $p$ is the dimension of constraints.

In this subsection, an LMI based scheme is proposed which satisfy simultaneously the constraints (19) and the inequality (9). Before it, we will introduce the concept of linear differential inclusion (LDI) of nonlinear systems.

Suppose that for all $x \in \mathcal{X}$, $u \in U$ and $w \in \mathcal{W}$, there exists a matrix $\Psi \in \mathcal{S} \subseteq \mathbb{R}^{n_x+n_u \times (n_x+n_u+n_w)}$ such that

$$
\begin{bmatrix} f(x, u, w) \\ h(x, u) \end{bmatrix} = \Psi \begin{bmatrix} x \\ u \\ w \end{bmatrix},
$$

then, $\mathcal{S}$ is a linear differential inclusion of the nonlinear system (1). Suppose that

$$
\mathcal{S} := \mathbb{C}_0 \left\{ \begin{bmatrix} A_1 & B_1 & G_1 \\ C_1 & D_1 & 0 \\ \vdots & \vdots & \vdots \\ A_N & B_N & G_N \\ C_N & D_N & 0 \end{bmatrix} \right\},
$$  \(20\)
then, $\Sigma$ is a polytopic linear differential inclusion (PLDI) of the given nonlinear system (1), and $[A_i \ B_i \ B_{wi}]$, $i \in Z_{[1,N]}$, is the vertex matrix of the set $\Sigma$, and $N$ is the number of vertex matrix.

If $f(0,0,0) = 0$ and

$$
\begin{bmatrix}
\frac{\partial f}{\partial x} & \frac{\partial f}{\partial u} & \frac{\partial f}{\partial w}
\end{bmatrix}
\in \Sigma
$$

for all $x \in \mathcal{X}$, $u \in \mathcal{U}$ and $w \in \mathcal{W}$, then, there exists a differential inclusion of the system, see e.g. [15].

The next lemma shows how to choose the terminal control law and the terminal set.

**Lemma 1:** Suppose that there exist a positive definite matrix $X \in \mathbb{R}^{n_x \times n_x}$, a non-square matrix $Y \in \mathbb{R}^{n_u \times n_x}$, and a scalars $\alpha > 0$ such that

$$
\begin{bmatrix}
\Theta & 0 & 0

\frac{\partial f}{\partial x} T_{i} & \Theta & 0 & 0

\frac{\partial f}{\partial u} & -\alpha I

\frac{\partial f}{\partial w} & C_i X + D_i Y & -\alpha I

\frac{\partial f}{\partial w} & X C_j T + Y T d_j & X
\end{bmatrix} \leq 0,
$$

(21)

for all $i \in Z_{[1,N]}$ and $j \in Z_{[1,p]}$, where $\Theta = (A_i X + B_i Y) T + A_i X + B_i Y$. Then, $u := Kx$ and $E(x) := x^T P x$ can be chosen as the terminal control and the terminal penalty, respectively, where $P := \alpha X^{-1}$ and $K := Y X^{-1}$.

**Idea of the proof:** The inequalities (21) and (22) guarantees that the nonlinear systems satisfies (9) and (2), respectively [16, 17].

**Remark 3.1:** In order to obtain the terminal set $\mathcal{X}_f$ as large as possible, one can solve the following optimization problem

$$
\text{maximize } (\det X)^{\frac{1}{n_x}}, \text{ s.t. (21), (22),}
$$

(22)

where $\alpha > 0$ and $X > 0$, to get the linear control law [18, 19]. The optimization problem is convex and can be solved by standard LMI solvers [15].

**IV. DISCUSSION ON NONLINEAR SYSTEMS WITH PERSISTENT BUT BOUNDED DISTURBANCES**

As we discussed, the robust invariant set plays an important role while the feasible solution is constructed. A control law is robust with respect to the disturbances or model-plant mismatches in a set is a necessary condition that the control law and the set are chosen as the terminal control law and the terminal set, respectively [8, 10, 11]. Thus, the min-max MPC scheme is not fit for energy bounded disturbances since they may drive the state system out of the given set. However, amplitude bounded disturbances can be considered.

Consider the exogenous disturbances $w \in \mathcal{W}$, where

$$
\mathcal{W} := \left\{ w \in \mathbb{R}^{n_w} \mid \|w\| \leq s \right\},
$$

(23)

and $s > 0$ is a scalar. Since $w$ is an amplitude bounded disturbance, $w \in \mathcal{W}_{2e}$. where

$$
\mathcal{W}_{2e} := \left\{ w \in \mathbb{R}^{n_w} \mid \int_{0}^{T} w(t)^T w(t) dt \leq \infty, \forall T \in [0, \infty) \right\}.
$$

(24)

In this subsection we will estimate the allowed upper bound of the exogenous disturbance that min-max MPC can deal with. Since the main argument is that the terminal control law is robust invariant in the terminal set with respect to the considered disturbances, we will mainly consider it from the aspect of the terminal control law and the terminal set.

Denote a set

$$
\mathcal{B}_0 := \left\{ x \in \mathbb{R}^{n_x} \mid x^T x \leq \frac{\alpha}{\lambda_{\text{max}}(P)} \right\}.
$$

Since $x^T P x \leq \lambda_{\text{max}}(P) x^T x$, $\mathcal{B}_0 \subseteq \mathcal{X}_f$.

**Lemma 2:** Suppose the linear feedback control law $Kx$ is chosen which is the solution of the standard $H_\infty$ control problem for the linearized system (11), and the disturbances satisfy

$$
\|w\|^2 \leq \frac{1}{\gamma_0^2} \inf_{x \in \mathcal{X}_f \cap \mathcal{B}_0} h(x, Kx)^T h(x, Kx).
$$

(25)

Then, the linear control law $Kx$, and the term $E(x) = \frac{1}{2} x^T P x$ and the ellipsoid $\mathcal{X}_f$ can serve as the terminal control law, the terminal penalty and the terminal set, respectively.

**Idea of the proof:** In the set $\mathcal{X}_f$, the nonlinear system (1) under the linear control law $Kx$ satisfies

$$
2 \dot{E}(x) \leq \gamma_0^2 \|w\|^2 - \|z\|^2.
$$

(26)

Due to $z = h(x, Kx)$, for all $x \in \mathcal{X}_f \cap \mathcal{B}_0$,

$$
2 \dot{E}(x) \leq \gamma_0^2 \|w\|^2 - h(x, Kx)^T h(x, Kx)
$$

$$
\leq \gamma_0^2 \|w\|^2 - \inf_{x \in \mathcal{X}_f \cap \mathcal{B}_0} h(x, Kx)^T h(x, Kx).
$$

Considering the inequality (25), we have $\dot{E}(x) \leq 0$. That is, the nonlinear system (1) under the linear control law $Kx$ is robust invariant in the set $\mathcal{X}_f$.

**Remark 3.2:** Lemma 2 gives an estimation on the upper bound of the disturbances which the min-max MPC can deal with, and the estimation does mainly depend on the chosen terminal set, terminal control law and terminal penalty matrix.

**V. CONCLUSION**

We proposed a scheme for the terminal set of min-max MPC with guaranteed $L_2$ performance, where structured feedback uncertainties are considered. Compared with the existing results, the existence of the terminal set is proved. Since, in general, a partial differential inequality is hard to solve, the requirement for solving an algebraic inequality instead is a great convenience in the design of min-max MPC control law. We also show that min-max MPC with $L_2$ performance can be used to deal with systems with bounded but small exogenous disturbances.

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